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UNIQUENESS OF PREDUALS OF CERTAIN BANACH SPACES†

BY
LEON BROWN AND TAKASHI ITO

ABSTRACT
In [1], the authors have shown the existence of non-quasireflexive Banach spaces having unique isomorphic preduals. In fact, certain James–Lindenstrauss' spaces have this property. In this paper it is shown that there are many such separable spaces. More precisely, there exist infinitely many different isomorphic types of James–Lindenstrauss' spaces which are non-quasireflexive and have unique isomorphic preduals.

A Banach space Y is said to be a predual of a Banach space X if Y^*, the dual of Y, is isomorphic (linearly homeomorphic) to X. A Banach space X is said to have a unique predual if X has a predual and all preduals are mutually isomorphic. It is not hard to see that quasireflexive spaces X (the quotient space X^{**}/X is finite dimensional) have unique preduals. In this paper, we are interested in uniqueness of preduals of non-quasireflexive spaces. In the first section, we consider spaces X such that X^{**}/X is isomorphic to $l_1(\Gamma)$. Theorem 1 is an improvement, and a generalization, of the result of [1].

In the second section we make a small modification of James–Lindenstrauss' spaces and show some of these spaces have a certain property; see Proposition 1.

In the third section, we consider spaces X such that X^{**}/X is reflexive. The idea used in proving Theorem 1 is used to obtain a necessary and sufficient condition for unique preduals; see Proposition 2. A sufficient condition (see Corollary) enables us to obtain a large class of James–Lindenstrauss' spaces with unique preduals; see Theorem 2 or Theorems 3 and 4 for a more abstract formulation.

1. The quotient space X^{**}/X is isomorphic to $l_1(\Gamma)$

THEOREM 1. *Let X be a Banach space satisfying the following two conditions:*

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a) *the quotient space* X^{**}/X *is isomorphic to* $l_1(\Gamma)$, *where* Γ *is any set*;

b) *if* X^* *does not contain a subspace isomorphic to* l_∞.

Then X^* *has a unique predual.*

Note that if X^* is separable, or more generally, if X^* is a weakly compactly generated Banach space, then X^* satisfies condition b).

The proof of Theorem 1 consists of five lemmas. Lemmas 1, 2, and 5 are the main steps of our proof. Lemma 2 and part of Lemma 1 are essentially given in our previous paper [1]. In the following we will always regard X and X^* as subspaces, respectively, of X^{**} and X^{***} in the canonical way. For a subspace A of X^{**}, $A^\perp$ denotes the annihilator of A in X^{***}, and for a subspace A of X^{***}, $A_\perp$ denotes the set of elements in X^{**} annihilated by A. According to Dixmier's observation [3], *all preduals* Y *of* X^* *can be identified (isomorphic) with all subspaces of* X^{**} *which are total over* X^* *and minimal with respect to the property of being total over* X^*. If Y is such a minimal total subspace of X^{**}, then we have the decomposition $X^{***} = X^* \oplus Y^\perp$, and by P_Y we denote the projection of X^{***} onto X^* with respect to this decomposition. In particular, the space X^{***} has the canonical decomposition; $X^{***} = X^* \oplus X^\perp$. An inner product $\langle u, v\rangle$ for $u \in X^{**}$ and $v \in X^{***}$ means the duality between X^{**} and X^{***}. When we refer to the wk* topology, we will always mean the X^{**} topology on X^{***}.

Since $l_1(\Gamma)$ has the "lifting property", the hypothesis X^{**}/X is isomorphic to $l_1(\Gamma)$ implies that X is complemented in X^{**}; see Köthe, p. 184 of [7]. Choose a subspace A so that $X^{**} = X \oplus A$, where A is isomorphic to $l_1(\Gamma)$. We will always regard $X^\perp$ as the dual space of A in the natural way although $X^\perp$ is not necessarily isometric to A^*. Finally, $K(X, A)$ denotes the set of all compact operators from X to A.

LEMMA 1. *There is a one to one correspondence between* $T \in K(X, A)$ *and minimal total subspaces* Y *of* X^{**} $(T \sim Y_T)$ *such that*:

i) $T^* = P_Y|_{X^\perp}$ *and* $Y^\perp = \{z - T^*z \mid z \in X^\perp\}$,

ii) $Y = \{x + a \mid Tx = a - T^{**}a,\ x \in X, a \in A\}$.

We denote the set given by ii) as Y_T.

PROOF. If a minimal total subspace Y of X^{**} is given, then the existence of $T \in K(X, A)$ satisfying i) and ii) is proven in [1] under the assumption of a countable Γ and a separable X^*. The argument given depends on Grothendieck's theorem: every operator from l_∞ into a separable Banach space is weakly compact. This theorem has been generalized in several ways. In particular

Rosenthal [10] shows that if X^* does not contain any isomorphic copy of l_∞ then every operator from $l_\infty(\Gamma)$ into X^* is weakly compact. Using this fact and an argument similar to the one on pp. 322 and 323 of [1], one can show the existence of $T \in K(X, A)$ satisfying i) and ii).

Conversely, if $T \in K(X, A)$ is given, then $E = \{z - T^*z \mid z \in X^\perp\}$ is wk*-closed in X^{***}. This can be seen easily by a standard argument as follows: It suffices to show $E \cap B_{X^{***}}$ ($B_{X^{***}}$ is the unit ball of X^{***}) is wk*-closed in X^{***}. If a net $\{z_\alpha - T^*z_\alpha\}$ in $E \cap B_{X^{***}}$ converges to $w \in X^{***}$ in the wk* topology, then $\{z_\alpha\}$ is a bounded net in $X^\perp$ because $\{z_\alpha\}$ is the image of $\{z_\alpha - T^*z_\alpha\}$ by a bounded projection defined by the decomposition $X^{***} = X^* \oplus X^\perp$. There is a subnet $\{z_{\alpha'}\}$ such that $z_{\alpha'} \xrightarrow{wk^*} z$, where $z \in X^\perp$ because $X^\perp$ is wk*-closed in X^{***}. Since T^* is compact, there exists a subnet $\{z_{\alpha''}\}$ of $\{z_{\alpha'}\}$ such that $T^*z_{\alpha''} \to y$ strongly. However $T^*z_{\alpha''} \to T^*z$ in the X topology on X^*. Therefore $T^*z = y$, $z_{\alpha''} - T^*z_{\alpha''} \xrightarrow{wk^*} z - T^*z$ and $w = z - T^*z \in E$, thus E is wk*-closed in X^{***}. It is easy to see that $X^* \oplus E = X^{***}$. Furthermore, with a computation and the fact that T^{**} maps $X^{**} = X \oplus A$ into A (T is compact) one sees that $E_\perp = Y_T$ (as given in ii)). Therefore, since E is wk*-closed, $E = (E_\perp)^\perp = Y_T^\perp$ and $X^* \oplus Y_T^\perp = X^{***}$. This implies that Y_T is a minimal total subspace of X^{**}. To show i), let $w \in X^\perp$, and we have $w = T^*w + (w - T^*w)$ with $T^*w \in X^*$ and $w - T^*w \in E = Y_T^\perp$ which implies $P_{Y_T}|_{X^\perp} = T^*$. Finally, the one to one correspondence is a consequence of i) and ii).

Lemma 2. *For any minimal total subspace Y of X^{**}, Y is complemented in X^{**} and X^{**}/Y is isomorphic to $l_1(\Gamma)$.*

Proof. This is proven in [1] and we outline the argument given (see p. 324). Suppose Y is given, $Y = Y_T$ for some compact operator $T \in K(X, A)$ (Lemma 1). Since T is compact, $I_A - T^{**}|_A$ becomes a Fredholm operator on A. Using this fact, we have decompositions of X and A as follows: $X = Z \oplus X_0$, where $Z = T^{-1}(\mathrm{Im}(I_A - T^{**}|_A))$ and X_0 is finite dimensional and $A = A_1 \oplus \mathrm{Ker}(I_A - T^{**}|_A)$, where A_1 is finite co-dimensional in A. Now one can show $Y_T \oplus X_0 \oplus A_1 = X^{**}$. Hence Y_T is complemented in X^{**} and X^{**}/Y_T is isomorphic to $X_0 \oplus A_1$, which is isomorphic to A.

Remark. In this lemma, to show that Y_T is complemented in X^{**}, our argument only depends on the fact that $I_A - T^{**}|_A$ is a Fredholm operator on A.

Lemma 3. *If $T \in K(X, A)$ has norm less than 1, then we have $Y_T \oplus A = X^{**}$. Therefore Y_T is isomorphic to X.*

PROOF. If $x + a = a_1$ where $Tx = a - T^{**}a$, $x \in X$, a, $a_1 \in A$, then $x = a - a_1 = 0$, $(X \cap A = \{0\})$ and $a - T^{**}a = 0$. Since $I_A - T^{**}|_A$ is invertible $(\| T^{**}|_A \| < 1)$, $a = 0$. Thus $Y_T \cap A = \{0\}$. To see that $Y_T \oplus A = X^{**}$, let $x^{**} = x + a$, $x \in X$, $a \in A$. Set $a_1 = (I_A - T^{**}|_A)^{-1}Tx$, then $x + a = x + a_1 + a - a_1$ with $x + a_1 \in Y_T$ and $a - a_1 \in A$.

LEMMA 4. *For T and S in $K(X, A)$ we have:*
i) $\| P_T \mid X^\perp - P_S \mid X^\perp \| \leqq \| T - S \|$,
ii) $\| P_S \mid Y_T^\perp \| \leqq \| P_T \mid X^\perp - P_S \mid X^\perp \| \, \| I_{X^{***}} - P_0 \|$,
where we denote P_{Y_T} by P_T for $T \in K(X, A)$.

PROOF. For $z \in X^\perp$

$$\| z \|_{A^*} = \sup_{\substack{a \in A \\ \|a\| \leqq 1}} |\langle a, z \rangle|.$$

Then we have $\| z \|_{A^*} \leqq \| z \|$ for $z \in X^\perp$. In fact $\| \cdot \|_{A^*}$ is equivalent to the original norm $\| \cdot \|$ on $X^\perp$ as a subspace of X^{***}. Since $T^* = P_T|_{X^\perp}$ and $S^* = P_S|_{X^\perp}$, we see that

$$\| T - S \| = \sup_{\substack{\|z\|_{A^*} \leqq 1 \\ z \in X^\perp}} \| T^*z - S^*z \| = \sup_{\substack{\|z\|_{A^*} \leqq 1 \\ z \in X^\perp}} \| P_T z - P_S z \|$$

$$\geqq \sup_{\substack{\|z\| \leqq 1 \\ z \in X^\perp}} \| P_T z - P_S z \| = \| P_T |_{X^\perp} - P_S |_{X^\perp} \|.$$

To see ii), we observe that $P_S P_T = P_T$ and $(I - P_T)\,(I - P_0) = I - P_T$. Hence $P_S(I - P_T) = P_S(I - P_T)\,(I - P_0)\,(I - P_T) = (P_S - P_T)\,(I - P_0)\,(I - P_T)$, which implies that P_S is equal to $(P_S - P_T)\,(I - P_0)$ on $Y_T^\perp$. Therefore $\| P_S |_{Y_T^\perp} \| \leqq \| P_S |_{X^\perp} - P_T |_{X^\perp} \| \, \| I - P_0 \|$.

LEMMA 5. *For any $T \in K(X, A)$ there is a positive number $\delta = \delta(T) > 0$ such that Y_S is isomorphic to Y_T if $\| T - S \| < \delta$ and $S \in K(X, A)$.*

PROOF. Fix $T \in K(X, A)$. By Lemma 1, Y_T is a minimal total subspace of X^{**}, hence we may regard X^*, X^{**} and X^{***} as Y_T^*, Y_T^{**}, and Y_T^{***} respectively. We denote the dual norm, the bidual norm and the triple dual norm of Y_T by $\| x^* \|_T$ for $x^* \in X^*$, $\| x^{**} \|_T$ for $x^{**} \in X^{**}$ and $\| x^{***} \|_T$ for $x^{***} \in X^{***}$ respectively. We can see that $\| \cdot \|_T \leqq \| \cdot \|$ on X^* and X^{***}, and $\| \cdot \|_T \geqq \| \cdot \|$ on X^{**}. By Lemma 2 we can choose a subspace B of X^{**} such that $X^{**} = Y_T \oplus B$ and B is isomorphic to $l_1(\Gamma)$. If $S \in K(X, A)$, then Y_S becomes a minimal total subspace of Y_T^*, hence we can apply Lemma 1 in this new situation. Then there is a compact operator R from Y_T to B such that

$R^* = P_S|_{Y_T^\perp}$, where we regard X^* as Y_T^* and $Y_S^\perp$ as B^* in the natural way. We want to show that the operator norm $\|R\|_T$ of R from $(Y_T, \|\cdot\|_T)$ to $(B, \|\cdot\|_T)$ is less than 1 if S is close enough to T.

Define $\|v\|_{B^*}$ on $Y_T^\perp$ by

$$\|v\|_{B^*} = \sup_{\substack{\|b\|_T \leq 1 \\ b \in B}} |\langle b, v\rangle|,$$

then $\|v\|_{B^*} \leqq \|v\|_T \leqq \|v\|$ for $v \in Y_T^\perp$. Since these three norms are equivalent on $Y_T^\perp$, there is $\alpha > 0$ such that

$$\|v\|_{B^*} \leqq \|v\| \leqq \alpha \|v\|_{B^*} \quad \text{for} \quad v \in Y_T^\perp.$$

Note that α depends upon only T and the choice of B.

$$\|R\|_T = \|R^*\|_T = \sup_{\substack{\|v\|_{B^*} \leq 1 \\ v \in Y_T^\perp}} \|P_S v\|_T$$

$$\leqq \sup_{\substack{\|v\| \leq \alpha \\ v \in Y_T^\perp}} \|P_S v\| = \alpha \|P_S \mid Y_T^\perp\|.$$

By Lemma 4, $\|R\|_T \leqq \alpha \|P_S|_{X^\perp} - P_T|_{X^\perp}\| \, \|I - P_0\| \leqq \alpha \|S - T\| \, \|I - P_0\|$. If $0 < \delta < \alpha \|I - P_0\|^{-1}$, $\|S - T\| < \delta$ and $S \in K(X, A)$, then $\|R\|_T < 1$. Therefore, Y_S has the subspace B as a complement in X^{**} (Lemma 3). Hence we can conclude that Y_S is isomorphic to Y_T.

Proof of Theorem 1. Define an equivalent relation: T is equivalent to S if T and S are in $K(X, A)$ and if Y_T is isomorphic to Y_S. By Lemma 5 each equivalent class is open with respect to the operator norm topology of $K(X, A)$, Thus it is open and closed. Since $K(X, A)$ is connected with respect to the norm topology, the space $K(X, A)$ must be one equivalent class. Thus the theorem is proved.

2. James–Lindenstrauss' spaces

A separable Banach space B is given. The construction of James–Lindenstrauss' spaces, which was initiated by James [4] and generalized by Lindenstrauss [8], is as follows: Choose a dense sequence $\{b_n\}$ on the unit sphere of B and an exponent r with $1 < r < \infty$. Define a Banach space E to be the set of all sequences $\{\xi_n\}$ of scalars such that

$$\|\{\xi_n\}\| = \sup \left(\sum_{j=1}^{k} \left\| \sum_{n_{j-1} < i \leq n_j} \xi_i b_i \right\|_B^r \right)^{1/r} < +\infty$$

where the sup is taken over all finite sets of integers with $0 = n_0 < n_1 < n_2 < \cdots < n_k$ and $k = 1, 2, \cdots$.

Let $e_n = \{\delta_{n,i}\}_{i=1}^{\infty}$ be the n-th unit vector of E. It is clear from the definition that $\{e_n\}$ form a monotone boundedly complete basis of E, hence E is isometric isomorphic to the dual of a Banach space $X (E = X^*)$, where X can be identified as the closed linear span of the biorthogonal functionals $\{f_n\}$ to the base $\{e_n\}$. It is also clear that we have a natural quotient map ϕ from E onto B by $\phi(\{\xi_n\}) = \sum_{n=1}^{\infty} \xi_n b_n$, hence ϕ^* is an isometric isomorphism from B^* into E^*. In [8] Lindenstrauss shows that for $r = 2$

$$E^* = X^{**} = X \oplus \phi(B^*).$$

The same argument holds for any r with $1 < r < \infty$. The space E is denoted by $JL(B, r)$. In this notation, we ignore the possibility that $JL(B, r)$ may depend on the choice of the sequence $\{b_n\}$.

We examine a property of some examples of James–Lindenstrauss' spaces.

PROPOSITION 1.

a) *Every operator from l_p to $JL(l_p, r)$ is compact if $1 < r < p < \infty$,*

b) *Every operator from L_p to $JL(L_p, r)$ is compact if $1 < r < \min\{2, p\}$, L_p denotes $L_p[0,1]$.*

PROOF. The proof will be similar to the argument used to prove compactness of operators from $L_p(\mu)$ to $L_r(\nu)$; see the appendix of Rosenthal [10].

We observe the following direct consequence of the definition of our norm in $JL(B, r)$. Suppose a sequence $\{x_n\}$ in $JL(B, r)$ is a normalized block basis of the natural basis $\{e_n\}$ of $JL(B, r)$, then for any positive integer n and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$, we have

$$\left(\sum_{k=1}^{n} |\alpha_k|^r\right)^{1/r} \leqq \left\| \sum_{k=1}^{n} \alpha_k x_k \right\|.$$

If $\{z_n\}$ is a normalized sequence which converges weakly to zero in $JL(B, r)$, we can choose a subsequence $\{z_{n_k}\}$ which is equivalent to a block basis; see Bessaga and Pelczynski [2]. Hence for a suitable constant K, we have

$$(*) \qquad \left(\sum_{k=1}^{m} |\alpha_k|^r\right)^{1/r} \leqq K \left\| \sum_{k=1}^{m} \alpha_k z_{n_k} \right\|$$

for any positive integer m and scalars $\alpha_1, \alpha_2, \cdots, \alpha_m$.

Secondly, suppose $\{y_n\}$ is a normalized sequence which converges weakly to zero in l_p or L_p for $1 < p < \infty$, then it follows from Bessaga and Pelczynski [2], Kadec and Pelczynski [5], and also observed by Rosenthal [10], that there is a subsequence $\{y_{n_k}\}$ such that for any positive integer m and scalars $\alpha_1, \alpha_2, \cdots, \alpha_m$

$$(**)\qquad \left\| \sum_{k=1}^{m} \alpha_k y_{n_k} \right\| \leqq C\left(\sum_{k=1}^{m} |\alpha_k|^{p'} \right)^{1/p'}$$

where $p' = p$ for the spaces l_p, $p > 1$ and L_p, $p \leqq 2$, $p' = p$ or 2 for the space L_p, $p > 2$ and C is a constant.

Suppose T is an operator from $l_p(L_p)$ to $JL(l_p, r)$ $(JL(L_p, r))$ which is not compact. Then there is a normalized sequence $\{y_n\}$ in $l_p(L_p)$ which converges weakly to zero and $\inf_n \| Ty_n \| = \delta > 0$. From the above two observations we can choose a subsequence $\{y_{n_k}\}$ such that $\{y_{n_k}\}$ satisfies (**) and $\{Ty_{n_k}\}$ satisfies (*). Then for any positive integer m and scalars $\alpha_1, \alpha_2, \cdots, \alpha_m$ we have

$$\begin{aligned}\left(\sum_{k=1}^{m} |\alpha_k|^{r} \right)^{1/r} &\leqq \frac{K}{\delta} \left\| \sum_{k=1}^{m} \alpha_k T y_{n_k} \right\| \\ &\leqq \frac{K\|T\|}{\delta} \left\| \sum_{k=1}^{m} \alpha_k y_{n_k} \right\| \\ &\leqq \frac{K\|T\|C}{\delta} \left(\sum_{k=1}^{m} |\alpha_k|^{p'} \right)^{1/p'}.\end{aligned}$$

Because $r < p'$, we have a contradiction which completes the proof.

3. The quotient space X^{**}/X is reflexive

Quasireflexive spaces X (X^{**}/X is finite dimensional) have unique preduals. However, it is not known whether the l_2-sum of a James Space J (J^{**}/J is one dimensional) has a unique predual. Or more generally, if $X^{**} = X \oplus A$, where X is a Banach space and A is infinite dimensional and reflexive, must X^* have a unique predual? Using the methods of Section 1, we can find necessary and sufficient conditions for such a space to have a unique predual.

PROPOSITION 2. *Let X be a Banach space such that X is complemented in X^{**} and X^{**}/X is reflexive, then X^* has a unique predual if and only if each minimal total subspace of X^{**} is complemented in X^{**}.*

PROOF. Choose A, a subspace of X^{**}, so that $X^{**} = X \oplus A$. The "only if" is easy to prove. Assume that X^* has a unique predual and Y is a minimal total subspace of X^{**}. If T is an isomorphism from X onto Y then T^{**} is an automorphism on X^{**}. $X^{**} = T^{**}(X) \oplus T^{**}(A) = Y \oplus T^{**}(A)$ and we have Y is complemented in X^{**}.

The "if" part requires three lemmas similar to Lemmas 1, 3 and 5. Let $B(X, A)$ denote the set of all bounded operators from X to A.

LEMMA 6. *There is a one to one correspondence between bounded operators $T \in B(X, A)$ and minimal total subspaces Y of X^{**} ($T \sim Y_T$) such that*

i) $T^* = P_Y|_{X^\perp}$ *and* $Y^\perp = \{z - T^*z \mid z \in X^\perp\}$,

ii) $Y = \{x + a \mid Tx = a - T^{**}a, x \in X, a \in A\} = Y_T$.

PROOF. If a minimal total subspace Y of X^{**} is given, let $S = P_Y|_{X^\perp}$; $S: X^\perp \to X^*$. Since $X^\perp$ can be regarded as the dual space of the reflexive space A, the weak topology on $X^\perp$ is equal to the A topology on $X^\perp$. Thus S is continuous if $X^\perp$ is given the A topology and X^* is given the X topology. Therefore there is an operator $T: X \to A$ such that $T^* = S = P_Y|_{X_\perp}$. It is not difficult to see that $Y^\perp = \{z - T^*z \mid z \in X^\perp\}$. $Y = \{x + a \mid x \in X,\ a \in A$ and $Tx = a - T^{**}a\}$ because an element $x + a$ in X^{**} ($x \in X, a \in A$) belongs to Y iff $x + a \in (Y^\perp)_\perp$ iff $\langle x + a, z - T^*z\rangle = 0$ for all $z \in X^\perp$ iff $\langle a - Tx - T^{**}a, z\rangle = 0$ for all $z \in X^\perp$ iff $a - Tx - T^{**}a = 0$ because $a - Tx - T^{**}a \in A$.

Conversely, if $T \in B(X, A)$, then $E = \{z - T^*z \mid z \in X^\perp\}$ is wk*-closed in X^{***}. This can be seen by showing $E \cap B_{X^{***}}$ is wk*-closed in X^{***}. Let $\{z_\alpha - T^*z_\alpha\}$ be a net which converges in the wk* topology to w. $\{z_\alpha\}$ is a bounded net because $\{z_\alpha\}$ is the image of *of* $\{z_\alpha - T^*z_\alpha\}$ by a bounded projection defined by the decomposition $X^{***} = X^* \oplus X^\perp$. Since $\{z_\alpha\}$ is bounded, there is a wk* limit point z of a subset $\{z_{\alpha'}\}$. $z \in X^\perp$ because $X^\perp$ is wk*-closed. Since A is reflexive, and the A topology on $X^\perp$ can be identified with the wk* topology of X^{***} restricted to $X^\perp$, $I_{X^\perp} - T^*$ is wk* $-$ wk* continuous from $X^\perp$ to X^{***}. Therefore $z - T^*z$ is a wk* limit point of the subset $\{z_{\alpha'} - T^*z_{\alpha'}\}$. Therefore, we have $w = z - T^*z$. The remaining argument is the same as in the proof of Lemma 1, except for the reason T^{**} maps X^{**} into A is that A is reflexive.

LEMMA 7. *If $T \in B(X, A)$ has norm less than* 1, *then we have $Y_T \oplus A = X^{**}$. Therefore Y_T is isomorphic to X.*

PROOF. The proof is identical to the proof of Lemma 3 except that one uses Lemma 6 instead of Lemma 2.

LEMMA 8. *For $T \in B(X, A)$ there is a positive number $\delta = \delta(T) > 0$ such that Y_S is isomorphic to Y_T if $\| T - S \| < \delta$.*

PROOF. The proof of this Lemma is identical to the proof of Lemmas 4 and 5, using Lemmas 6 and 7 and the hypothesis that each minimal total subspace Y of

X^{**} is complemented (complement must be reflexive) instead of Lemmas 1, 2 and 3.

PROOF OF PROPOSITION 2. The proof is completed using Lemma 8 as we proved Theorem 1 using Lemma 5.

From Proposition 2 we get sufficient conditions for unique preduals.

COROLLARY. *Let X be a Banach space such that* $X^{**} = X \oplus A$ *and A is reflexive. If every operator from X into A is a compact operator then* X^* has a unique predual. Or more generally, for every operator T from X into A, the operator $T^{**}|_A$ is a strictly singular operator on A then X^* has a unique predual (see T. Kato [6] for definition and properties of strictly singular operators).

PROOF. If $T^{**}|_A$ is compact or more generally strictly singular then $I_A - T^{**}|_A$ is a Fredholm operator. The proof is completed using the remark to Lemma 2 and Proposition 2.

THEOREM 2. *The following James–Lindenstrauss' spaces have unique preduals*:

a) $JL(l_p, r)$ *for* $1 < r < p < \infty$,

b) $JL(L_p, r)$ *for* $1 < r < \min\{2, p\}$,

c) $JL(c_0, r)$ *for* $1 < r < \infty$.

PROOF. a) and b) is a direct consequence of Proposition 1 and the Corollary of Proposition 2. c) follows from Theorem 1.

Note that if $JL(B_1, r_1)$ and $JL(B_2, r_2)$ are two spaces mentioned in a), b), or c) of Theorem 2 and B_1 is not isomorphic to B_2 then $JL(B_1, r_1)$ is not isomorphic to $JL(B_2, r_2)$. This follows from the fact that these spaces have unique preduals.

One can generalize a) of Theorem 2 as follows:

THEOREM 3. *Let X be a Banach space satisfying the following two conditions*:

a) $X^{**} = X \oplus A$ *and A is isomorphic to* l_q *with* $1 < q < \infty$,

b) X^* *does not contain a subspace isomorphic to* l_p *with* $1/p + 1/q = 1$, *then* X^* *has a unique predual.*

PROOF. Let T be an operator from X to A, then T^* is a strictly singular operator from A^* to X^*. This follows from condition b) and the fact that every infinite dimensional subspace of l_p contains a subspace isomorphic to l_p; see Pelczynski [9]. Since l_p is subprojective[†], a theorem of Whitley (see [12]) implies

[†] Every infinite dimensional subspace contains an infinite dimensional subspace complemented in the entire space.

that T^{**} is a strictly singular operator from X^{**} to A. Thus $T^{**}|_A$ is a strictly singular operator on A (in fact compact) and the Corollary to Proposition 2 implies X^* has a unique predual.

In a completely similar manner one can prove

THEOREM 4. *Let X be a Banach space satisfying the following three conditions:*

a) $X^{**} = X \oplus A$ *and A is reflexive,*

b) *A is subprojective,*

c) *X^* and A^* are totally incomparable*[†], *then X^* has a unique predual.*

Note that Theorem 3 is a special case of Theorem 4 and if A is isomorphic to L_q, $2 \leqq q < \infty$ then we have a generalization of a part of b), in Theorem 2.

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WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN, 48202, USA

WAYNE STATE UNIVERSITY

AND

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS, 61801, USA

[†] Two Banach spaces are totally incomparable if they have no isomorphic subspaces of infinite dimension.